Since $f(a+x) \equiv (a+x)^{n+1} - a^{n+1} \equiv \binom{n+1}{1}a^n x + \binom{n+1}{2}a^{n-1}x^2 + \cdots + \binom{n+1}{i+1}a^{n-i}x^{i+1} + \cdots + x^{n+1}$, and again,

$$f(a+x) \equiv f(a) + xf'(a) + \frac{x^2}{\underline{|2}}f''(a) + \cdots + \frac{x^{i+1}}{(i+1)!}f^{(i+1)}(a) + \cdots + \frac{x^{n+1}}{(n+1)!}(n+1)!,$$

we have, on comparison of coefficients,

$$\frac{1}{(i+1)!}f^{(i+1)}(a) = \binom{n+1}{i+1}a^{n-1} = \frac{(n+1)!}{(i+1)!(n-i)!}a^{n-1},$$

$$f^{(i+1)}(a) = (i+1)\phi^{(i)}(a) = (i+1)S_{i,\,n} \cdot a^{n-1} = \frac{(n+1)!}{(n-i)!}a^{n-1},$$

so that

$$S_{i,\,n} = \frac{1}{i+1} \cdot \frac{(n+1)!}{(n-i)!},$$

and $(n+1)!/(n-i)!$ is the last term of $S_{i+1,\,n+1} = \sum_{k=1}^{k=n-i+1} (i+k)!/(k-1)!$.

Also solved by HORACE OLSON.

457. Proposed by FRANK IRWIN, University of California.

If a be any number prime to m and m/a be developed as a continued fraction,

$$\frac{m}{a} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k},$$

with $a_1 \neq 0$, then there will exist a number b such that

$$\frac{m}{b} = a_k + \frac{1}{a_{k-1}} + \cdots + \frac{1}{a_2} + \frac{1}{a_1}.$$

Show that $ab \equiv \pm 1$ (mod. m) and determine the sign.

SOLUTION BY THE PROPOSER.

Write m and a as functions of the a's: $m = [a_1, a_2, \cdots a_k]$, $a = [a_2, a_3, \cdots a_k]$, using the *Gaussische Klammer* notation, see BACHMANN, *Niedere Zahlentheorie*, vol. 1, p. 104; or, with a different symbol, $K(a_1, a_2, \cdots a_k)$, CHRYSTAL's *Algebra*, part II, 2d ed., p. 495.

Similarly, $b = [a_{k-1}, \cdots a_2, a_1]$, and if p_{k-1}/q_{k-1} be the next to last convergent to m/a, $p_{k-1} = [a_1, a_2, \cdots a_{k-1}]$. But, by an elementary property of these expressions, $[a_1, a_2, \cdots a_{k-1}] = [a_{k-1}, \cdots a_2, a_1)$; that is, $p_{k-1} = b$.

Again, since p_{k-1}/q_{k-1}, and m/a are successive convergents, $mq_{k-1} - ap_{k-1} = (-1)^k$; that is, $mq_{k-1} - ab = (-1)^k$, or $ab \equiv (-1)^{k-1}$ (mod. m).

GEOMETRY.

483. Proposed by LAENAS G. WELD, Pullman, Illinois.

A circle is inscribed in a triangle. In each of the three spandrils exterior to the circle another circle is inscribed; in the remaining spandrils three other circles; and so on ad infinitum. Show that the sum of the areas of these circles is given by the formula:

$$\Sigma = \frac{\pi}{4} \cdot \frac{\Delta^2}{S^2}\left[\frac{1}{\sin(A/2)} + \frac{1}{\sin(B/2)} + \frac{1}{\sin(C/2)} - 2 + \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right].$$

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let A, B, C and a, b, c denote the angles and sides opposite these angles of the given triangle. Since the center of the inscribed circle is at the intersection of the bisectors of the angles of the triangle, the centers of the circles inscribed in the spandrils are on these bisectors. Let R be the radius of the inscribed circle and R_1, R_2, $\cdots$ R_n the radius of the circles whose centers are A_1, A_2, $\cdots$ A_n, respectively.

From the triangle AOB we can easily show that

$$AO = \frac{c \sin B/2}{\sin (A+B)/2} \quad \text{and} \quad R = \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2};$$

and since

$$AA_1 = AO - A_1O = AO - R - R_1, \qquad AA_1 = \frac{R_1}{\sin A/2}.$$

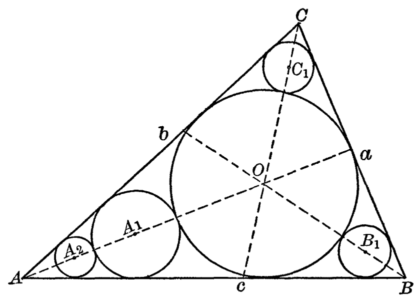
Hence, we have

$$\frac{R_1}{\sin A/2} = \frac{c \sin B/2}{\sin (A+B)/2} - \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2} - R_1.$$

Solving for R_1 we get

$$R_1 = \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2} \cdot \frac{1 - \sin A/2}{1 + \sin A/2}.$$

In the same way we can prove from $AA_2 = AO - A_2O$ that



$$R_2 = \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2} \cdot \left(\frac{1 - \sin A/2}{1 + \sin A/2}\right)^2,$$

and finally, for circles on OA, $R_n = R(F_a)^n$; and similarly for circles on OB, $R_n = R(F_b)^n$; for circles on OC, $R_n = R(F_c)^n$, where

$$F_a = \frac{1 - \sin A/2}{1 + \sin A/2}, \qquad F_b = \frac{1 - \sin B/2}{1 + \sin B/2}, \qquad F_c = \frac{1 - \sin C/2}{1 + \sin C/2}.$$

The sum Σ of the areas of all these circles will then be

$$\Sigma = \pi R^2 + \pi R^2(F_a)^2 + \pi R^2(F_a)^4 + \pi R^2(F_a)^6 + \cdots \pi R^2(F_a)^{2n}$$
$$+ \pi R^2(F_b)^2 + \pi R^2(F_b)^4 + \cdots \pi R^2(F_b)^{2n} + \pi R^2(F_c)^2 + \pi R^2(F_c)^4 + \cdots \pi R^2(F_c)^{2n};$$

or,

$$\Sigma = \pi R^2[1 + (F_a)^2 + (F_a)^4 + \cdots (F_a)^{2n} + 1 + (F_b)^2 + (F_b)^4$$
$$+ \cdots (F_b)^{2n} + 1 + (F_c)^2 + (F_c)^4 + \cdots (F_c)^{2n} - 2].$$

Adding the terms of each of these geometric series by the usual formula, we have

$$\Sigma = \pi R^2 \left[\frac{(F_a)^{2n} - 1}{(F_a)^2 - 1} + \frac{(F_b)^{2n} - 1}{(F_b)^2 - 1} + \frac{(F_c)^{2n} - 1}{(F_c)^2 - 1} - 2 \right].$$

If n approaches infinity, we have

$$\Sigma = \pi R^2 \left[\frac{1}{1 - (F_a)^2} + \frac{1}{1 - (F_b)^2} + \frac{1}{1 - (F_c)^2} - 2 \right].$$

Replacing the values of F_a, F_b, and F_c we get

$$\Sigma = \frac{\pi R^2}{4} \left[\frac{(1 + \sin A/2)^2}{\sin A/2} + \frac{(1 + \sin B/2)^2}{\sin B/2} + \frac{(1 + \sin C/2)^2}{\sin C/2} - 2 \right].$$

Let Δ be the area of the given triangle, then $\Delta = \frac{1}{2}(a + b + c)R$ or $R = \Delta S$ where $S = \frac{1}{2}(a + b + c)$.

Substituting the value of R in Σ, expanding the binomials, and reducing we finally get,

$$\Sigma = \frac{\pi}{4}\cdot\frac{\Delta^2}{S^2}\left[\frac{1}{\sin A/2}+\frac{1}{\sin B/2}+\frac{1}{\sin C/2}-2+\sin A/2+\sin B/2+\sin C/2\right].$$

Also solved by F. R. Morris, Horace Olson, G. W. Hartwell, J. W. Clawson, Paul Capron, and J. W. Cromwell.

484. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that when spheres of uniform size are packed in the closest possible manner there is, in the interior of the mass, about 26 per cent. of voids.

Solution by Laenas G. Weld, Pullman, Ills.

The given space may be divided into (equal) rhombic dodecahedrons. Spheres inscribed to these will be packed in the closest possible manner, since each has contact with twelve equal spheres. Now, the rhombic dodecahedron may be conceived as follows: Place a cube and a regular octahedron in such relation that each of the twelve edges of either figure is bisected at right angles by an edge of the other; then join the extremities of the edges so related, thus forming twelve rhombs, which are the faces of the figure in question. Each rhomb has for one of its diagonals an edge of the cube and for the other the corresponding edge of the octahedron. Taking the edge of the octahedron as 1, that of the cube related to it as above is equal to $\frac{1}{2}\sqrt{2}$. Moreover the radius of the inscribed sphere is equal to $\frac{1}{2}$, which is also the altitude of each of the twelve rhombic pyramids into which the dodecahedron may be resolved. The area of the base of each of these pyramids is one half the product of its diagonals, which is equal to $\frac{1}{4}\sqrt{2}$; whence the volume of the dodecahedron is

$$D = 12(\tfrac{1}{3}\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}\sqrt{2}) = \tfrac{1}{2}\sqrt{2}.$$

The volume of the sphere is $S = \pi/6$.

The ratio S/D is equal to the ratio of the aggregate volume of the spheres to that of the space in which they are packed, since this space is completely filled by the circumscribing rhombic dodecahedrons. This ratio is $0.7405-$. The voids, therefore, occupy $0.2595+$ of the space, or about 26 per cent.

Also solved with slightly different results by Paul Capron, G. Paaswell, J. W. Clawson, and Herbert N. Carleton.

CALCULUS.

400. Proposed by H. S. UHLER, Yale University.

The axis of a prism whose right section is a regular polygon of apothem a and n sides passes through the center of a sphere of radius R. Show that, in general, the volume may be expressed by the formula:

$$V = \tfrac{4}{3}\pi R^3 + \tfrac{2}{3}a^2n\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}$$
$$+\tfrac{1}{3}an(3R^2-a^2)\sin^{-1}\left[\frac{2a\left(R^2-a^2\sec^2\dfrac{\pi}{n}\right)^{1/2}\tan\dfrac{\pi}{n}}{R^2-a^2}\right] - \tfrac{4}{3}nR^3\sin^{-1}\left[\frac{R\sin\dfrac{\pi}{n}}{(R^2-a^2)^{1/2}}\right].$$

Also discuss the special cases when $a = R\cos(\pi/n)$ and when $n = \infty$.

Solution by R. K. Morley, Worcester, Mass.

Using cylindrical coördinates

$$V = 4n\int_0^{\pi/n}\int_0^{a\sec\theta}\int_0^{\sqrt{R^2-}}\rho\,dz\,d\rho\,d\theta = -\tfrac{4}{3}n\int_0^{\pi/n}(R^2-a^2\sec^2\theta)^{3/2}d\theta + \tfrac{4}{3}n\int_0^{\pi/n}R^3d\theta$$

$$= -\tfrac{4}{3}n\int_0^{\pi/n}(R^2-a^2\sec^2\theta)(R^2-a^2\sec^2\theta)^{1/2}d\theta + \tfrac{4}{3}\pi R^3.$$